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On ordinary generalized geometric–arithmetic index

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ABSTRACT

The ordinary generalized geometric–arithmetic index of graphs is introduced and some properties especially lower and upper bounds in terms of other graph invariants and topological indices are obtained.

1. Introduction

Topological indices are numerical parameters of a graph which are invariant under graph isomorphisms. They play a significant role in chemistry, pharmacology, physics, etc. [1]. For a graph G, with vertex set $V(G)$ and edge set $E(G)$, motivated by the definition of Randić connectivity index [2,3], Vukičević and Furtula in [4] proposed the geometric–arithmetic index (GA for short) of G as:

$$\mathrm{GA}(G)=\sum_{uv\in E(G)}\frac{2\sqrt{d_ud_v}}{(d_u+d_v)},$$

where d_u denotes the degree of vertex $u\in V(G)$. Also the Randić connectivity index ($R(G)$) and sum-connectivity index ($\chi(G)$) of G are defined as:

$$R(G)=\sum_{uv\in E(G)}(d_ud_v)^{-\frac{1}{2}}$$

$$\chi(G)=\sum_{uv\in E(G)}(d_u+d_v)^{-\frac{1}{2}}.$$

In 1998, Bollobàs and Erdös in [5] generalized the Randić index by replacing $-\frac{1}{2}$ with any real number k, which is called the general Randić index:

$$R_k(G)=\sum_{uv\in E(G)}(d_ud_v)^k.$$

Similarly, Zhou and Trinajstić, in [6], introduced the ordinary sum connectivity

$$\chi_k(G)=\sum_{uv\in E(G)}(d_u+d_v)^k.$$

They obtained some properties, especially lower and upper bounds in terms of the other graph invariants, of this index.

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A class of geometric–arithmetic topological indices was defined in [7] as follows:

$$\mathrm{GA}_{\mathrm{general}}(G) = \sum_{uv\in E(G)} \frac{\sqrt{Q_u Q_v}}{\frac{1}{2}(Q_u + Q_v)},$$

where Q_u is some quantity that in a unique manner can be associated with the vertex u of the graph G. Some classes of GA indices have been studied in [7–10]. It is natural that we consider the ordinary geometric–arithmetic index of G. It is defined for each positive real number k as:

$$\mathrm{OGA}_k(G) = \sum_{uv\in E(G)} \left[\frac{\sqrt{4d_u d_v}}{d_u + d_v}\right]^k = \sum_{uv\in E(G)} \frac{(4d_u d_v)^{\frac{k}{2}}}{(d_u + d_v)^k}.$$

In this paper, we focus our attention to this index and its main properties including lower and upper bounds.

2. Bounds for the ordinary geometric–arithmetic index

We recall that if $f''(x) > 0$, for the real value function $f(x)$ defined on an interval, then $f(x)$ is a strictly convex function. Jensen's inequality shows that if $x_1, x_2, \ldots, x_k$ are arbitrary real numbers, then $f\left(\sum_{i=1}^{k} \frac{x_i}{k}\right) \leq \frac{1}{k}\sum_{i=1}^{k} f(x_i)$.

If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are n-tuples of real numbers satisfying

$$0 \leq m_1 \leq a_i \leq M_1 \quad \text{and} \quad 0 \leq m_2 \leq b_i \leq M_2,$$

for $i = 1, 2, \ldots, n$, then by Ozeki's inequality [11]:

$$\sum_{i=1}^{m} a_i^2 \sum_{i=1}^{m} b_i^2 - \left[\sum_{i=1}^{m} a_i b_i\right]^2 \leq \frac{m^2}{4}(M_1 M_2 - m_1 m_2)^2.$$

Consider two n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ of real numbers, let $(w_1, \ldots, w_n)$ be an n-tuple of positive real numbers. Then the following inequality (Hölder's inequality with weight) holds:

$$\sum_{k=1}^{n} w_k |a_k b_k| \leq \left(\sum_{k=1}^{n} w_k |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} w_k |b_k|^q\right)^{\frac{1}{q}},$$

for each pair p, q of positive integers such that $\frac{1}{p} + \frac{1}{q} = 1$.

In the following theorem, we obtain a relation between $\mathrm{OGA}_k(G)$ and $\mathrm{OGA}_{2k}(G)$.

Theorem 1. *Let G be a simple graph with n vertices and m edges. Then*

$$m\mathrm{OGA}_{2k}(G) - \frac{m^2}{4}\left(1 - \frac{2^k (n-1)^{\frac{k}{2}}}{n^k}\right)^2 \leq [\mathrm{OGA}_k(G)]^2 \leq m\mathrm{OGA}_{2k}(G).$$

Proof. Let $w_{uv} = (2d_u d_v)^k$, $a_{uv} = \frac{1}{(d_u + d_v)^k}$ and $b_{uv} = \frac{1}{(d_u d_v)^{\frac{k}{2}}}$. By Hölder's inequality, $p = q = 2$,

$$\left[\sum_{uv\in E(G)} w_{uv} a_{uv} b_{uv}\right]^2 \leq \left[\sum_{uv\in E(G)} w_{uv} a_{uv}^2\right]\left[\sum_{uv\in E(G)} w_{uv} b_{uv}^2\right].$$

But

$$w_{uv} a_{uv} b_{uv} = (2d_u d_v)^k \frac{1}{(d_u + d_v)^k} \frac{1}{(d_u d_v)^{\frac{k}{2}}} = \frac{(4d_u d_v)^{\frac{k}{2}}}{(d_u + d_v)^k},$$

$$w_{uv} a_{uv}^2 = (2d_u d_v)^k \frac{1}{(d_u + d_v)^{2k}},$$

$$w_{uv} b_{uv}^2 = (2d_u d_v)^k \frac{1}{(d_u d_v)^k} = 2^k.$$

Therefore

$$[\mathrm{OGA}_k(G)]^2 \leq \left[\frac{\mathrm{OGA}_{2k}(G)}{2^k}\right][m2^k] = m\mathrm{OGA}_{2k}(G).$$

Without loss of generality, we may assume that $d_u \leq d_v$. Denote $x = \frac{d_u}{d_v}$ and note that $\frac{1}{n-1} \leq x \leq 1$. Now we consider $f(x) = \frac{x^{\frac{k}{2}}}{(1+x)^k}$. Simple differential calculation shows that $f'(x) = \frac{k}{2}\frac{x^{\frac{k}{2}}(1-x)}{x(1+x)^{k+1}}$. Therefore f is ascending on the interval $\left(\frac{1}{n-1}, 1\right]$, hence it achieves minimum at $x = \frac{1}{n-1}$ and maximum at $x = 1$. Let $a_{uv} = \frac{(d_u d_v)^{\frac{k}{2}}}{(d_u+d_v)^k}$ and $b_{uv} = 2^k$. Then $m_1 := f\left(\frac{1}{n-1}\right) \leq a_{uv} \leq \frac{1}{2^k} := M_1$ and $m_2 := 2^k \leq b_{uv} \leq 2^k := M_2$. We have

$$\sum_{uv\in E(G)} a_{uv}^2 = \sum_{uv\in E(G)} \frac{(d_u d_v)^k}{(d_u+d_v)^{2k}} = \frac{1}{4^k}\mathrm{OGA}_{2k}(G),$$
$$\sum_{uv\in E(G)} b_{uv}^2 = m4^k,$$
$$\left[\sum_{uv\in E(G)} a_{uv} b_{uv}\right]^2 = \left[\mathrm{OGA}_k(G)\right]^2.$$

Therefore by Ozeki's inequality

$$m\mathrm{OGA}_{2k}(G) - \frac{m^2}{4}\left(1 - \frac{2^k(n-1)^{\frac{k}{2}}}{n^k}\right)^2 \leq [\mathrm{OGA}_k(G)]^2. \quad \square$$

Theorem 2. *Let G be a graph with n vertices and m edges. Then*

$$\frac{m2^k(n-1)^{\frac{k}{2}}}{n^k} \leq \mathrm{OGA}_k(G) \leq m.$$

The lower bound is achieved if and only if G is a star and the upper bound is achieved if and only if G is a regular graph.

Proof.

$$\mathrm{OGA}_k(G) = \sum_{uv\in E(G)} \frac{(4d_u d_v)^{\frac{k}{2}}}{(d_u+d_v)^k} = \sum_{uv\in E(G)} \left[\frac{\sqrt{d_u d_v}}{\frac{d_u+d_v}{2}}\right]^k \leq \sum_{uv\in E(G)} 1^k = m.$$

The upper bound is achieved if and only if for each edge uv of G, $d_u = d_v$ and therefore G must be a regular graph. Also in the proof of Theorem 1, we proved that if $f(x) = \frac{x^{\frac{k}{2}}}{(1+x)^k}$, then $f\left(\frac{1}{n-1}\right) \leq \frac{(d_u d_v)^{\frac{k}{2}}}{(d_u+d_v)^k}$. Therefore $\mathrm{OGA}_k(G) \geq m2^k f\left(\frac{1}{n-1}\right) = \frac{m2^k(n-1)^{\frac{k}{2}}}{n^k}$. Moreover, the equality is obtained if and only if the graph has $n-1$ edges each connecting vertices of degrees 1 and n. The only such graph is a star. $\square$

Theorem 3. *Let G be a simple graph with m vertices, maximum degree Δ and minimum degree δ. Then*

$$m\left(\frac{2}{\delta+\Delta}\right)^k (\delta\Delta)^{\frac{k}{2}} \leq \mathrm{OGA}_k(G)$$

and the equality holds if and only if for each edge of G, its end vertices have degrees δ and Δ.

Proof. Let $f(x) = \frac{x^{\frac{k}{2}}}{(1+x)^k}$. In the proof of Theorem 1, we proved that $f(x) = \frac{x^{\frac{k}{2}}}{(1+x)^k}$ is an increasing function on $\left(\frac{1}{n-1}, 1\right]$. Assume that $uv \in E(G)$ and $d_v \leq d_u$. Then $\mathrm{OGA}_k(G) = \sum_{uv\in E(G)} 2^k f\left(\frac{d_v}{d_u}\right)$. Note that

$$f\left(\frac{d_v}{d_u}\right) \geq f\left(\frac{\delta}{\Delta}\right) = \left(\frac{1}{\delta+\Delta}\right)^k (\delta\Delta)^{\frac{k}{2}},$$

and the equality holds if and only if $d_v = \delta$ and $d_u = \Delta$. $\square$

Theorem 4. *Let T be a tree with $n \geq 3$ vertices, then*

$$\mathrm{OGA}_k(T) \leq 2\frac{2^{\frac{3k}{2}}}{3^k} + (n-3).$$

The upper bound is achieved if and only if T is a path.

Proof. Let u be a pendant vertex in T and v be its only neighbor. Then

$$\frac{(d_u d_v)^{\frac{k}{2}}}{(d_u + d_v)^k} = \frac{(d_v)^{\frac{k}{2}}}{(1 + d_v)^k}.$$

Consider the function $\frac{x^{\frac{k}{2}}}{(1+x)^k}$. We have $f'(x) = \frac{k}{2}\frac{x^{\frac{k}{2}}(1-x)}{x(1+x)^{k+1}}$. Therefore f is descending on the segment $[2, n-1]$. Hence

$$\frac{(d_u d_v)^{\frac{k}{2}}}{(d_u + d_v)^k} = \frac{(d_v)^{\frac{k}{2}}}{(1 + d_v)^k} \le \frac{2^{\frac{k}{2}}}{3^k}.$$

Let l denote the number of leaves (vertices of degree 1) in T. Since, every tree has at least two leaves and $\left(\frac{2^{\frac{3k}{2}}}{3^k} - 1\right) \le 0$, it holds:

$$\mathrm{OGA}_k(T) \le \frac{2^{\frac{3k}{2}}}{3^k}.l + (n-1-l).1 \le 2\frac{2^{\frac{3k}{2}}}{3^k} + (n-3).$$

Moreover, the equality holds if and only if the tree has exactly two edges connecting vertices of degrees 1 and 2, and all other edges connect vertices of the same degree. The only such graph is a path. □

Corollary 5. *Let G be a simple graph with n vertices and m edges. Then*

$$m\left(\frac{2}{n}\right)^k (n-1)^{\frac{k}{2}} \le \mathrm{OGA}_k(G),$$

and the equality holds if and only if G is a star.

Proof. Consider the proof of Theorem 3. Then for each $uv \in E(G)$, $\frac{1}{n-1} \le \frac{d_v}{d_u}$. Therefore the equality holds if and only if for each $uv \in E(G)$ we have $\frac{1}{n-1} = \frac{d_v}{d_u}$. Since the degree of each vertex of G is at most $n-1$, G must be a star. □

Theorem 6. *Let G be a graph with $m \ge 1$ edges. Then if $k > 1$ then*

$$\mathrm{OGA}_k(G) \ge \frac{\mathrm{OGA}_1(G)^k}{m^{k-1}}.$$

If $0 < k < 1$, then the inequality is reversed.

Proof. If $k > 1$ then x^k is a strictly convex function and thus (by Jensen's inequality):

$$\left[\frac{1}{m}\mathrm{OGA}_1(G)\right]^k = \left[\frac{1}{m}\sum_{uv\in E(G)}\frac{2\sqrt{d_u d_v}}{d_u + d_v}\right]^k \le \frac{1}{m}\sum_{uv\in E(G)}\left[\frac{2\sqrt{d_u d_v}}{d_u + d_v}\right]^k = \frac{1}{m}\mathrm{OGA}_k(G).$$

Hence $\mathrm{OGA}_k(G) \ge [\mathrm{OGA}_1(G)]^k m^{1-k}$. For $k < 1$, using the concavity of x^k completes the proof. □

Corollary 7. *Let G be a simple connected graph with n vertices and m edges. Then if $k > 1$, then*

$$\mathrm{OGA}_k(G) \ge \frac{1}{m^{k-1}}\left[\frac{2(n-1)^{\frac{3}{2}}}{n}\right]^k.$$

If $0 < k < 1$, then the inequality is reversed. The equality holds if and only if G is a star.

Proof. By [4], $\frac{2(n-1)^{\frac{3}{2}}}{n} \le \mathrm{OGA}_1(G)$ and the equality holds if and only if G is a star. □

Corollary 8. *Let G be a triangle-free graph with a simple connected graph with n vertices and $m \ge 1$ edges. Then*

$$\mathrm{OGA}_k(G) \ge \frac{1}{m^{k-1}}\left[\frac{4m^2}{n^2}\right]^k,$$

with the equality holds if and only if G is a regular complete bipartite graph.

Proof. By Proposition 8 of [8], $\frac{4m^2}{n^2} \le \mathrm{OGA}_1(G)$ and the equality holds if and only if G is a regular complete graph. □

Theorem 9. *Let G be a molecular graph. Then*

$$\mathrm{OGA}_k(G) \le 2^k R_{-\frac{k}{4}}(G).$$

Proof. Let $g(x, y) = (x + y)^4 - (xy)^3$. Then for each $x, y \in \{1, 2, 3, 4\}$ we have $g(x, y) \geq 0$. Therefore for each k, $\frac{(xy)^k}{(x+y)^{2k}} \leq \frac{1}{(xy)^{\frac{k}{2}}}$. Since G is a molecular graph, for each $u \in V(G)$ we have $d_u \in \{1, 2, 3, 4\}$. Hence

$$\mathrm{OGA}_{2k}(G) = \sum_{uv\in E(G)} \frac{4^k(d_ud_v)^k}{(d_u + d_v)^{2k}} \leq \sum_{uv\in E(G)} \frac{4^k}{(d_ud_v)^{\frac{k}{2}}} = 4^k R_{-\frac{k}{2}}(G). \quad \square$$

Theorem 10. *Let G be a graph with m edges and minimum degree δ. Then*

$$\mathrm{OGA}_1(G) \leq \frac{1}{\delta}\sqrt{mR_1(G)}.$$

The equality holds if and only if G is a regular graph.

Proof. We have

$$\begin{aligned}\mathrm{OGA}_1(G) &= \sum_{uv\in E(G)} \frac{2\sqrt{d_ud_v}}{(d_u + d_v)} \leq \sum_{uv\in E(G)} \frac{\sqrt{d_ud_v}}{\delta} \leq \sqrt{\sum_{uv\in E(G)} \left(\frac{1}{\delta}\right)^2 \sum_{uv\in E(G)} d_ud_v} \\ &= \frac{1}{\delta}\sqrt{mR_1(G)}. \quad \square\end{aligned}$$

Theorem 11. *Let G be a graph with m edges and n vertices and maximum degree Δ. Then*

$$\mathrm{OGA}_1(G) \geq \sqrt{\frac{R_1(G)}{\Delta^2} + \frac{4\Delta}{(1+\Delta)^2}m(m-1)}.$$

The equality holds if and only if G is a regular graph.

Proof. We have

$$\begin{aligned}[\mathrm{OGA}_1(G)]^2 &= 4\sum_{uv\in E(G)} \frac{d_ud_v}{(d_u + d_v)^2} + 2\sum_{uv\neq \acute{u}\acute{v}\in E(G)} \frac{2\sqrt{d_ud_v}}{d_u + d_v}\frac{2\sqrt{d_{\acute{u}}d_{\acute{v}}}}{d_{\acute{u}} + d_{\acute{v}}} \\ &\geq 4\sum_{uv\in E(G)} \frac{d_ud_v}{4\Delta^2} + 2\sum_{uv\neq \acute{u}\acute{v}\in E(G)} \frac{2\sqrt{\Delta}}{(1+\Delta)}\frac{2\sqrt{\Delta}}{(1+\Delta)} \\ &= \frac{R_1}{\Delta^2} + \frac{4\Delta}{(1+\Delta)^2}m(m-1),\end{aligned}$$

which completes the proof. $\square$

Theorem 12. *Let G be a triangle-free graph with n vertices. Then*

$$\mathrm{OGA}_k(G) \geq \left(\frac{2}{n}\right)^k R_{\frac{k}{2}}.$$

Proof. Since G is a triangle-free graph, for each $uv \in E(G)$, $d_u + d_v \leq n$ and thus we have

$$\mathrm{OGA}_k(G) = \sum_{uv\in E(G)} \frac{(4d_ud_v)^{\frac{k}{2}}}{(d_u + d_v)^k} \geq \sum_{uv\in E(G)} \frac{(4d_ud_v)^{\frac{k}{2}}}{n^k} = \left(\frac{2}{n}\right)^k R_{\frac{k}{2}}(G).$$

If the equality holds in the above inequality, then $d_u + d_v = n$ for any $uv \in E(G)$, G has no isolated vertices and every component of G is regular, and thus $d_u = \frac{n}{2}$ for any $u \in V(G)$, implying that G is the regular complete bipartite graph since G is triangle free. Conversely, it is easily seen that the lower bound for $\mathrm{OGA}_k(G)$ is attained for the regular complete bipartite graph. $\square$

Nordhaus and Gaddum [12] gave bounds for the sum of the chromatic numbers of a graph and its complement. Nordhaus–Gaddum-type results for many graph invariants are known. Here we give a Nordhaus–Gaddum-type result for the OGA index.

Theorem 13. *Let G be a graph with m edges and n vertices. Then*

$$\frac{2^k(n-1)^{\frac{k}{2}}}{n^k}\frac{n(n-1)}{2} \leq \mathrm{OGA}_k(G) + \mathrm{OGA}_k(\bar{G}) \leq \frac{n(n-1)}{2}.$$

The lower bound is achieved if and only if G is a star and the upper bound is achieved if and only if G is a regular graph.

Proof. By Theorem 2

$$\frac{m2^k(n-1)^{\frac{k}{2}}}{n^k} \leq \mathrm{OGA}_k(G) \leq m.$$

Since $\bar{G}$ has $\frac{n(n-1)}{2} - m$ edges and n vertices we can obtain the desired result. □

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